

Shifted Character Sums with Multiplicative Coefficients

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Abstract. Let $f(n)$ be a multiplicative function satisfying $|f(n)| \leq 1$, q ($\leq N^2$) be a prime number and a be an integer with $(a, q) = 1$, χ be a non-principal Dirichlet character modulo q . In this paper, we shall prove that

$$\sum_{n \leq N} f(n) \chi(n+a) \ll \frac{N}{q^{\frac{1}{4}}} \log \log(6N) + q^{\frac{1}{4}} N^{\frac{1}{2}} \log(6N) + \frac{N}{\sqrt{\log \log(6N)}}.$$

We shall also prove that

$$\begin{aligned} \sum_{n \leq N} f(n) \chi(n+a_1) \cdots \chi(n+a_t) &\ll \frac{N}{q^{\frac{1}{4}}} \log \log(6N) \\ &+ q^{\frac{1}{4}} N^{\frac{1}{2}} \log(6N) + \frac{N}{\sqrt{\log \log(6N)}}, \end{aligned}$$

where $t \geq 2$, a_1, \dots, a_t are pairwise distinct integers modulo q .

1. Introduction

Let q be a prime number, a be an integer with $(a, q) = 1$, χ be a non-principal Dirichlet character modulo q .

Since the 1930s, I. M. Vinogradov had begun the study on character sums over shifted primes

$$\sum_{p \leq N} \chi(p+a),$$

and obtained deep results [8, 9], where p runs through prime numbers. His best known result is a nontrivial estimate for the range $N^\varepsilon \leq q \leq N^{\frac{4}{3}-\varepsilon}$, where ε is a sufficiently small positive constant, which lies deeper than the direct consequence of Generalized Riemann Hypothesis. Later, Karatsuba [4] widen the range to $N^\varepsilon \leq q \leq N^{2-\varepsilon}$, where Burgess's method was applied.

For the Möbius function $\mu(n)$, one can get same results on sums

$$\sum_{n \leq N} \mu(n) \chi(n+a)$$

as that on sums over shifted primes.

In this paper, we consider the general sum

$$\sum_{n \leq N} f(n) \chi(n+a), \quad (1.1)$$

where $f(n)$ is a multiplicative function satisfying $|f(n)| \leq 1$. We shall apply the method in Section 2 in [2], which is called as the finite version of Vinogradov's inequality, to give a nontrivial estimate for the sum (1.1) when q is in a suitable range.

Theorem 1. Let $f(n)$ be a multiplicative function satisfying $|f(n)| \leq 1$, q ($\leq N^2$) be a prime number and a be an integer with $(a, q) = 1$, χ be a non-principal Dirichlet character modulo q . Then we have

$$\begin{aligned} \sum_{n \leq N} f(n) \chi(n+a) &\ll \frac{N}{q^{\frac{1}{4}}} \log \log(6N) \\ &+ q^{\frac{1}{4}} N^{\frac{1}{2}} \log(6N) + \frac{N}{\sqrt{\log \log(6N)}}. \end{aligned} \quad (1.2)$$

Remark. The estimate in (1.2) is nontrivial for

$$(\log \log(6N))^{4+\varepsilon} \ll q \ll \frac{N^2}{(\log(6N))^{4+\varepsilon}},$$

which should be compared with the conjectural nontrivial range as indicated by Karatsuba [4, p.325].

In the same way as Theorem 1, we shall prove the following

Theorem 2. We assume that $f(n)$, q , χ same as in Theorem 1. For $t \geq 2$ and pairwise distinct integers a_1, \dots, a_t modulo q , we have

$$\begin{aligned} \sum_{n \leq N} f(n) \chi(n+a_1) \cdots \chi(n+a_t) &\ll \frac{N}{q^{\frac{1}{4}}} \log \log(6N) \\ &+ q^{\frac{1}{4}} N^{\frac{1}{2}} \log(6N) + \frac{N}{\sqrt{\log \log(6N)}}, \end{aligned} \quad (1.3)$$

where implied constant depends on t .

Remarks. 1) Taking $f = \mu$ to be the Möbius function in Theorem 2, we obtain an example for the Möbius Randomness Law. Such an example answers, in a special case, a problematic issue posed by P. Sarnak, see [7, p.3].

2) The $t = 2$ case of Theorem 2 corresponds to Karatsuba [5]. While for any $t \geq 3$, our result should be compared with a conditional result of Karatsuba [6], which relies on a conjectural upper bound for a kind of character sums in two variables.

Throughout this paper, we assume that N is sufficiently large and set

$$\begin{aligned} d_0 &= \sqrt{\log \log(6N)}, \quad D_0 = e^{d_0} = \exp(\sqrt{\log \log(6N)}), \\ d_1 &= d_0^2 = \log \log(6N), \quad D_1 = e^{d_1} = \log(6N). \end{aligned} \quad (1.4)$$

Let p, q denote prime numbers, ε be a sufficiently small positive constant.

2. The proof of Theorem 1

Lemma 1. Let $f(n)$ be a multiplicative function satisfying $|f(n)| \leq 1$, q be a prime number, a be an integer, χ be a Dirichlet character modulo q . d_0, d_1 are defined as in (1.4). Then we have

$$\begin{aligned} & \sum_{n \leq N} f(n) \chi(n+a) \\ & \ll \sum_{r=[d_0]+1}^{[d_1]-1} \sum_{y \leq \frac{N}{e^r}} \left| \sum_{\substack{e^r \leq p < e^{r+1} \\ (p,q)=1 \\ p \leq \frac{N}{y}}} f(p) \chi(py+a) \right| + \frac{N}{\sqrt{\log \log(6N)}} + \frac{N}{q}. \end{aligned} \quad (2.1)$$

Proof. We have

$$\sum_{n \leq N} f(n) \chi(n+a) = \sum_{\substack{n \leq N \\ (n,q)=1}} f(n) \chi(n+a) + \sum_{\substack{n \leq N \\ (n,q)>1}} f(n) \chi(n+a)$$

and

$$\sum_{\substack{n \leq N \\ (n,q)>1}} f(n) \chi(n+a) \ll \sum_{\substack{n \leq N \\ q|n}} 1 \ll \frac{N}{q} + 1.$$

By the discussion in Section 2 of [3], the idea of which was adopted from [2] with some modification, we get

$$\begin{aligned} \sum_{\substack{n \leq N \\ (n,q)=1}} f(n) \chi(n+a) & \ll \sum_{r=[d_0]+1}^{[d_1]-1} \sum_{y \leq \frac{N}{e^r}} \left| \sum_{\substack{e^r \leq p < e^{r+1} \\ (p,q)=1 \\ p \leq \frac{N}{y}}} f(p) \chi(py+a) \right| \\ & + \frac{N}{\sqrt{\log \log(6N)}}. \end{aligned}$$

Hence, the conclusion of Lemma 1 follows.

Lemma 2. Let q be a prime number, χ_1, \dots, χ_r be Dirichlet characters modulo q , at least one of which is non-principal. Let $f(X) \in \mathbb{F}_q[X]$ be an arbitrary polynomial of degree d . Then for pairwise distinct $a_1, \dots, a_r \in \mathbb{F}_q$, we have

$$\left| \sum_{x \in \mathbb{F}_q} \chi_1(x + a_1) \cdots \chi_r(x + a_r) e\left(\frac{f(x)}{q}\right) \right| \leq (r + d)q^{\frac{1}{2}}.$$

This is Lemma 17 in [1].

Lemma 3. Let q be a prime number, χ be a non-principal Dirichlet character modulo q . Then for an arbitrary integer h with $1 \leq h \leq q$ and distinct $s, t \in \mathbb{F}_q$,

$$\sum_{x=1}^h \chi\left(\frac{x+s}{x+t}\right) = O(q^{\frac{1}{2}} \log q)$$

holds true. Here we write $\frac{1}{n}$ as the multiplicative inverse of n such that $\frac{1}{n} \cdot n \equiv 1 \pmod{q}$ and appoint $\frac{1}{0} = 0$.

This is Lemma 18 in [1].

Lemma 4. Let q be a prime number, χ be a non-principal Dirichlet character modulo q , $(a, q) = 1$. Then for two primes p_1, p_2 with $(p_1, q) = (p_2, q) = 1$, $p_1 \not\equiv p_2 \pmod{q}$, we have

$$\sum_{X < y \leq Z} \chi\left(\frac{p_1 y + a}{p_2 y + a}\right) \ll \frac{Z - X}{q} \sqrt{q} + \sqrt{q} \log q.$$

Proof. By Lemmas 2 and 3, we have

$$\begin{aligned} \sum_{X < y \leq Z} \chi\left(\frac{p_1 y + a}{p_2 y + a}\right) &= \chi(p_1) \bar{\chi}(p_2) \sum_{X < y \leq Z} \chi\left(\frac{y + a \bar{p}_1}{y + a \bar{p}_2}\right) \\ &\ll \frac{Z - X}{q} \sqrt{q} + \sqrt{q} \log q. \end{aligned}$$

Let

$$Y = \frac{N}{e^r}. \tag{2.2}$$

We shall estimate the sum

$$\sum_1 = \sum_{y \leq Y} \left| \sum_{\substack{e^r \leq p < e^{r+1} \\ (p, q) = 1 \\ p \leq \frac{N}{y}}} f(p) \chi(py + a) \right|. \tag{2.3}$$

By Cauchy's inequality, we have

$$\sum_1 \leq Y^{\frac{1}{2}} \left(\sum_{y \leq Y} \left| \sum_{\substack{e^r \leq p < e^{r+1} \\ (p, q) = 1 \\ p \leq \frac{N}{y}}} f(p) \chi(py + a) \right|^2 \right)^{\frac{1}{2}}.$$

An application of Lemma 4 to

$$\sum_2 = \sum_{y \leq Y} \left| \sum_{\substack{e^r \leq p < e^{r+1} \\ (p, q) = 1 \\ p \leq \frac{N}{y}}} f(p) \chi(py + a) \right|^2$$

produces

$$\begin{aligned} \sum_2 &= \sum_{y \leq Y} \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q) = 1 \\ p_1 \leq \frac{N}{y}}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q) = 1 \\ p_2 \leq \frac{N}{y}}} f(p_1) \overline{f(p_2)} \chi\left(\frac{p_1 y + a}{p_2 y + a}\right) \\ &= \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q) = 1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q) = 1}} f(p_1) \overline{f(p_2)} \sum_{\substack{y \leq Y \\ y \leq \frac{N}{\max(p_1, p_2)}}} \chi\left(\frac{p_1 y + a}{p_2 y + a}\right) \\ &\ll \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q) = 1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q) = 1}} \left| \sum_{\substack{y \leq Y \\ y \leq \frac{N}{\max(p_1, p_2)}}} \chi\left(\frac{p_1 y + a}{p_2 y + a}\right) \right| \\ &\ll \sum_{e^r \leq p_1 < e^{r+1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ p_2 \equiv p_1 \pmod{q}}} Y \\ &\quad + \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q) = 1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q) = 1 \\ p_2 \not\equiv p_1 \pmod{q}}} \left| \sum_{\substack{y \leq Y \\ y \leq \frac{N}{\max(p_1, p_2)}}} \chi\left(\frac{p_1 y + a}{p_2 y + a}\right) \right| \\ &\ll Y e^r \left(\frac{e^r}{q} + 1 \right) + \sum_{e^r \leq p_1 < e^{r+1}} \sum_{e^r \leq p_2 < e^{r+1}} \left(\frac{Y}{\sqrt{q}} + \sqrt{q} \log q \right) \\ &\ll Y e^r + \frac{Y e^{2r}}{\sqrt{q}} + \sqrt{q} (\log q) e^{2r}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_1 &\ll Y^{\frac{1}{2}} \left(\frac{Y e^{2r}}{\sqrt{q}} + Y e^r + \sqrt{q} (\log q) e^{2r} \right)^{\frac{1}{2}} \\ &\ll \frac{Y e^r}{q^{\frac{1}{4}}} + Y e^{\frac{r}{2}} + q^{\frac{1}{4}} (\log q)^{\frac{1}{2}} Y^{\frac{1}{2}} e^r \\ &\ll \frac{N}{q^{\frac{1}{4}}} + \frac{N}{e^{\frac{r}{2}}} + q^{\frac{1}{4}} (\log q)^{\frac{1}{2}} N^{\frac{1}{2}} e^{\frac{r}{2}}. \end{aligned}$$

Applying this estimate to (2.1), we get

$$\begin{aligned}
& \sum_{n \leq N} f(n) \chi(n+a) \\
& \ll \sum_{r=[d_0]+1}^{[d_1]-1} \left(\frac{N}{q^{\frac{1}{4}}} + \frac{N}{e^{\frac{r}{2}}} + q^{\frac{1}{4}} (\log q)^{\frac{1}{2}} N^{\frac{1}{2}} e^{\frac{r}{2}} \right) + \frac{N}{\sqrt{\log \log(6N)}} + \frac{N}{q} \\
& \ll \frac{N}{q^{\frac{1}{4}}} \log \log(6N) + \frac{N}{\exp(\frac{1}{2} \sqrt{\log \log(6N)})} \\
& \quad + q^{\frac{1}{4}} (\log q)^{\frac{1}{2}} N^{\frac{1}{2}} (\log(6N))^{\frac{1}{2}} + \frac{N}{\sqrt{\log \log(6N)}} \\
& \ll \frac{N}{q^{\frac{1}{4}}} \log \log(6N) + q^{\frac{1}{4}} N^{\frac{1}{2}} \log(6N) + \frac{N}{\sqrt{\log \log(6N)}}.
\end{aligned}$$

So far the proof of Theorem 1 is complete.

3. The proof of Theorem 2

Lemma 5. Let q be a prime number, χ be a non-principal Dirichlet character modulo q . Then for an arbitrary integer h with $1 \leq h \leq q$ and pairwise distinct $a_1, \dots, a_t, b_1, \dots, b_t (t \geq 2) \in \mathbb{F}_q$,

$$\sum_{x=1}^h \chi \left(\frac{(x+a_1) \cdots (x+a_t)}{(x+b_1) \cdots (x+b_t)} \right) = O(q^{\frac{1}{2}} \log q)$$

holds true, where the implied constant depends on t .

Proof. It comes from Lemma 2 in the same way as Lemma 3.

Lemma 6. Let q be a prime number, χ be a non-principal Dirichlet character modulo q . Assume that $t \geq 2$, $(a_1, q) = \cdots = (a_t, q) = 1$, a_1, \dots, a_t are pairwise distinct modulo q and that primes p_1, p_2 satisfy $(p_1, q) = (p_2, q) = 1$, $p_1 \not\equiv p_2 \pmod{q}$. Then if $p_2 \not\equiv \overline{a_i} a_j p_1 \pmod{q}$ ($1 \leq i, j \leq t$), we have

$$\sum_{X < y \leq Z} \chi \left(\frac{(p_1 y + a_1) \cdots (p_1 y + a_t)}{(p_2 y + a_1) \cdots (p_2 y + a_t)} \right) \ll \frac{Z-X}{q} \sqrt{q} + \sqrt{q} \log q.$$

If $p_2 \equiv \overline{a_i} a_j p_1 \pmod{q}$ for some (i, j) , we have trivial bound

$$\sum_{X < y \leq Z} \chi \left(\frac{(p_1 y + a_1) \cdots (p_1 y + a_t)}{(p_2 y + a_1) \cdots (p_2 y + a_t)} \right) \ll Z - X + 1.$$

Proof. It can be proved in the same way as Lemma 4.

We can prove Theorem 2 in the same way as Theorem 1, by discussing the case in which one of a_i is equivalent to $0 \pmod{q}$ and the case in which no a_i is equivalent to $0 \pmod{q}$ respectively.

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